Local public transportation firms: the relevance of scale and scope economies in the provision of urban and intercity bus transit

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Local public transportation firms: the relevance of scale and scope economies in the provision of urban and intercity bus transit

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Abstract

Using a panel of local public transport companies, four different cost specifications are compared. The standard translog and the generalized translog specifications give unreliable estimates with respect to scope economies because of the degenerate behaviour of such functions when outputs are set to zero. The separable quadratic and the composite models allow the direct handling of zero outputs and they better fit the data. Moderate global scope economies are estimated from the preferred specification (0.2%), but splitting global scope economies into its two components, it is found that large fixed cost savings can be obtained from joint production (6.3%). Density economies and modest scale economies are also detected for the median firm in the sample.

Keywords: scope and scale economies, transport companies, cost function
JEL: C33, L25, L33, L92

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1. Introduction

Only few papers analysed scope economies in the local public transport (LPT) industry.

Farsi et al. (2007) use a sample of Swiss multi mode transport companies, supplying transport services using trolley – bus, motor – bus and / or tramway systems. They estimate a quadratic cost function and they show that both economies of scale and economies of scope exist.

Fraquelli et al. (2004a) study a sample of Italian municipal public transit companies supplying intercity and / or urban services. Imposing a translog functional form to their model for variable operating costs, they find (a) that companies operating in the intercity sector have lower costs than urban firms and (b) that companies supplying both urban and intercity services have lower costs than specialised firms. Their results, however, cannot in principle be compared to scope economies since they are based on the inclusion of a dummy variable for the type of activity in the cost specification. The derivation of scope economies encompasses the use of more information as they are based on the estimated cost structure, allowing for an out- of - sample prediction of total costs associated to zero output levels.

A growing literature exists on scope economies and in particular on the choice of the functional form that is better suited for the study of multi products technologies. Most of the studies are however applied to the banking sector (Pulley and Braunstein, 1992; McKillop et al., 1996; Adams et al., 2004) and utilities different from transport industries (Bloch et al., 2001; Fraquelli et al., 2004b; Fraquelli et al., 2005;).

This paper represents one of the few papers trying to tackle the issues of measuring scope economies in the transport industry and to evaluate different cost functional forms to this end.

In the next section we are going to present the four specification forms for the cost function. The standard translog model, the generalized translog, the separable quadratic function and the composite model are estimated using a sample of Italian local public transport companies. The sample is described in section 3: an unbalanced panel of 67 firms observed over the period 1998-2004, supplying urban services (3 companies) or intercity connections (33 firms) or both activities (31 companies).

Section 4 reports the estimation results and the density, scale and scope economies obtained from the different cost specifications. The standard translog and the generalized translog specifications give unreliable estimates because of the degenerate behaviour of such functions when outputs are set to zero. The separable quadratic and the composite models allow the direct handling of zero outputs and they better fit the data. Moderate global scope economies are estimated from the preferred composite specification (0.2%), but splitting global scope economies into its two components, it is found that large fixed cost savings can be obtained from joint production (6.3%). Density economies and modest scale economies are also detected for the median firm in the sample.

Section 5 concludes and an appendix reports the main technical details about the adopted formulas and the obtained results.

2. Model specification

2.1 The cost function

Our aim is to estimate a cost function given the observations for a set of local public transport (LPT) firms, obtaining some insights on scope and scale economies within the industry. We concentrate on the total cost function where two outputs (urban transit and intercity transit) and two inputs (labour price and capital price) enter. Moreover the characteristics of the served area, such as the network length associated to each output, also enter the cost structure.

Literature on scope economies showed the unreliable results that are obtained when a standard translog specification is adopted. Pulley and Braunstein (1992) and Pulley and Humphrey (1993) discuss the desirable properties of the composite specification with respect to other functional forms. McKillop et al. (1996) and Piacenza and Vannoni (2004) draw similar conclusions comparing the estimated scope economies from a set of different cost specifications using data on Japanese banks and Italian utilities respectively.

Our strategy is to estimate four different cost specifications and compare their ability to fit the data and to give sensible estimates for the scope economies.

Given the definition of a Box-Cox (1964) transformation for a generic variable y, represented by a subscript in parenthesis:

$$y^{(\pi)} = (y^{\pi} - 1)/\pi \qquad if \qquad \pi \neq 0$$
$$= \ln(y) \qquad if \qquad \pi = 0$$

we then follow Pulley and Braunstein (1992), who introduce the functional form the four specifications we are going to estimate are nested in:

$$\ln(C) = \left\{ \ln \left[\exp\left(\left(\alpha_0 + \sum_i \alpha_i q_i^{(\pi)} + \frac{1}{2} \sum_{i,j} \alpha_{ij} q_i^{(\pi)} q_j^{(\pi)} + \sum_{i,r} \alpha_{ir} q_i^{(\pi)} \ln p_r + \right. \right. \right. \right.$$
$$\left. \left. \left. + \sum_i \delta_i n_i^{(\pi)} + \frac{1}{2} \sum_{i,j} \delta_{ij} n_i^{(\pi)} n_j^{(\pi)} + \sum_{i,r} \delta_{ir} n_i^{(\pi)} \ln p_r + \frac{1}{2} \sum_{i,j} \lambda_{ij} q_i^{(\pi)} n_j^{(\pi)} \right)^{(\tau)} \right) \right] + \qquad (1)$$
$$\left. \left[\sum_r \beta_r \ln p_r + \frac{1}{2} \sum_{r,k} \beta_{rk} \ln p_r \ln p_k \right] \right\}$$

where C is the total cost, q_i is output i, $i=U, I$ for urban and intercity outputs respectively; n_i is the network length associated to the provision of output i and p_r is the price of input r, $r=L, K$ for labour and capital prices respectively.

Imposing different restrictions to the parameters, the four models follow[1]:

- *standard translog:* $\tau=1, \pi=0$
- *generalized translog:* $\tau=1$
- *separable quadratic:* $\tau=0, \pi=1, \alpha_{ir} = \delta_{ir} = 0$
- *composite function:* $\tau=0, \pi=1$

The main drawback with the widely adopted standard translog specification is its poor behaviour when outputs are equal to zero. While the translog specification is a flexible approximation to the true cost function in a point and its immediate neighbourhood[2], which is usually chosen to be the mean or median point of the data in the sample, its behaviour is unpredictable outside that point and in particular in the neighbourhood of

---

[1] Parameter restrictions for symmetry and linear homogeneity in input prices are also imposed (see section 3 on data description). These restrictions are identical for all specifications (see Pulley and Humphrey, 1993):

Symmetry: $\alpha_{ij} = \alpha_{ji}, \delta_{ij} = \delta_{ji}, \beta_{rk} = \beta_{kr}$ *and Linear Homogeneity:* $\Sigma_r\alpha_{ir} = \Sigma_r\delta_{ir} = 0$ *for all i;* $\Sigma_r\beta_r=1$ *;* $\Sigma_k\beta_{rk}=0$ *for all k*

[2] The standard translog can be interpreted as a second order Taylor series expansion of any arbitrary twice differentiable cost function at a given point. See Christensen et al. (1971) who first introduced the translog model and Diewert (1974) and Diewert and Wales (1987) for the definition of flexible functional forms.

zero output levels. The translog cost function has a degenerate limiting behaviour since total costs equal zero or infinity when one output approaches zero (see Roller, 1990) and in general the size of the region around zero where the translog is badly behaved is unknown ex ante and it actually depends on the parameter estimates (see Pulley and Humphrey, 1993). It follows that scope economies / diseconomies tend to be large because stand alone costs (i.e. costs along the axis where one output is set to zero) are too small (large scope diseconomies follow, see the formula for scope economies in the next subsection) or too large (large scope economies are obtained).

The generalized translog cost function was suggested by Caves et al. (1980) in order to admit zero output levels. The output variables are transformed using the Box-Cox (1964) transformation, instead of the logarithmic form, and an additional parameter needs to be estimated (the π parameter in equation (1) above). When this parameter equals zero, the translog specification is obtained, but it can be shown that also for small values of it (below one) the behaviour of the generalized translog model is not different from that of the translog model in the neighbourhood of zero output levels, leading to unsatisfactory scope economies estimates.

The separable quadratic model and the composite model have quadratic structures in outputs and log-quadratic structures in input prices: while the separable quadratic specification imposes strong separability among inputs and outputs, the composite model avoids this restriction allowing for the interaction among outputs and input prices. The quadratic output structure was first recommended by Baumol et al. (1982) when examining scope economies because this form allows for the direct handling of zero outputs, without any need for substitutions or transformations as in the two translog models.

The econometric model follows by adding a stochastic error term to the specification of the cost function, i.e.:

$$\ln(C)_{ft} = g(\mathbf{q}_{ft}, \mathbf{n}_{ft}, \mathbf{p}_{ft}) + u_{ft} \qquad (2)$$

where *g()* is now the generic functional form for the cost structure; (***q***, ***n***, ***p***) are the vectors representing all outputs, networks' lengths and input prices, u is the error term and subscripts f and t denote the panel structure of the data since f is for firms, $f=1,\ldots,F$ and t for time, $t=1,\ldots,T$. We can impose an error component structure to the error term, i.e. $u_{ft} = \mu_f + \nu_{ft}$ where μ_f is the unobserved time invariant individual specific effect, while ν_{ft} is the remainder error term that varies across individuals and time.

It is important to control for the likely econometric issues that can arise when a model is estimated using longitudinal data (see, among the others, Carey, 1997, Garcia and Thomas, 2001, for cost functions; Greene, 2001, for nonlinear models). Given the high nonlinearities of most of the estimated specifications, we are going to make a number of assumptions with respect to the correlation between individual (firm) specific unobservables and the independent variables[3]. The fixed effects model that allows for potential correlation among the individual specific effects and the regressors cannot be easily extended to nonlinear models. In general when trying to estimate the full set of individual specific effects (e.g. like in Least Squares Dummy Variable, LSDV, models where the full set of individuals' dummy variables are included into the specification and estimated together with the other parameters of the model), the incidental parameter problem arises and all estimates are inconsistent[4]. A unique solution to the problem does not exist and many model - specific solutions have been suggested (see Cameron and Trivedi, 2005, ch. 23 for a survey).

The approach we are going to follow in the empirical application is to assume a pooled nonlinear model estimated via nonlinear least squares where inference is based on panel robust standard errors, i.e. where we control for conditional heteroscedasticity and conditional correlation among observations from the same firm f across time. Consistency is ensured by the validity of the assumption of absence of correlation among individual effects and the set of independent variables that enter the model in equation (2): $E[\mu_f | \mathbf{q}_{ft}, \mathbf{n}_{ft}, \mathbf{p}_{ft}] = 0$.

[3] Our estimation strategy is not going to consider the simultaneous estimation of the cost function and the input shares equations as they are obtained from the Shephard's Lemma. The main advantage from the estimation of a system of simultaneous equations using Zellner's Seemingly Unrelated Regression method is a gain in efficiency, that is not a major problem in our dataset given the satisfactory precision of most of the estimated parameters.

[4] When T is small and N is large (as in our dataset), the incidental parameter problem stems from the inconsistency of the estimated fixed effects, that are based on a very small number of observations (the T time series). The econometric difficulty is that this inconsistency "propagates" to all the parameters of the model.

2.2 Economies of scope and size returns

Given the multi output nature of the considered cost function, it is possible to evaluate the presence and the importance of scope economies. Scope economies exist if the costs for one single firm providing both urban and intercity services are lower than those of two bus companies specialized in only one of these services. For a production technology with *m* outputs, following Baumol et al. (1982), scope economies are measured by evaluating the costs of specialized versus joint production:

$$
\begin{aligned}
&SCOPE = (C[q_1,0,\dots,0;\overline{p}] + C[0,q_2,\dots,0;\overline{p}] + \dots \\
&+ C[0,0,\dots,q_m;\overline{p}] - C[q_1,q_2,\dots,q_m;\overline{p}]) / C[q_1,q_2,\dots,q_m;\overline{p}]
\end{aligned}
$$

where for a generic output *i*, $C(0,0,\dots, q_i,\dots,0;\overline{p})$ is the cost associated to the production of output *i* only, constraining all the other *m-1* outputs to zero, while $C(q_1, q_2,\dots, q_m;\overline{p})$ is the joint cost of producing all *m* outputs; *p* is the vector of input prices that are kept constant at a given level in the computation of the different cost magnitudes. In our empirical specification we are going to deal with two outputs (urban and intercity bus services) and we are going to keep the input prices fixed at their sample median level. Scope economies are detected if the value of *SCOPE*>0, while diseconomies arise if *SCOPE*<0.

We are also going to consider a different measure for scope economies: quasi scope economies (*QSCOPE*). In this case the cost savings are assessed with respect to a quasi – specialised production of the different outputs (see the discussion in Pulley and Humphrey, 1993):

$$
\begin{aligned}
&QSCOPE = (C[(1-(m-1)\varepsilon)q_1, \varepsilon q_2,\dots,\varepsilon q_m;\overline{p}] + C[\varepsilon q_1,(1-(m-1)\varepsilon)q_2,\dots,\varepsilon q_m;\overline{p}] + \dots \\
&+ C[\varepsilon q_1,\varepsilon q_2,\dots,(1-(m-1)\varepsilon)q_m;\overline{p}] - C[q_1,q_2,\dots,q_m;\overline{p}]) / C[q_1,q_2,\dots,q_m;\overline{p}]
\end{aligned}
$$

where ε is the share of other outputs' production and it ranges between *0* and *1/m*. When $\varepsilon = 0$, quasi scope economies are identical to global scope economies (*QSCOPE* = *SCOPE*) while for increasing values of ε production is less and less specialised, implying different output mix. When $\varepsilon = 1/m$, each output is produced in equal proportion and *QSCOPE* becomes a measure of the difference between the costs associated to *m* firms each producing *1/m^th^* of each output and a single firm supplying the whole production of all outputs.

Size returns measure how the average cost changes when the size of the firm increases. In particular economies of size are present if the average cost decreases as a consequence of the enhanced size.

Since the seminal contribution by Caves et al. (1984), the inclusion of a variable describing the network size in the estimation of the transport cost function, makes it necessary to distinguish between returns to density and returns to scale. Returns to density (*DENSITY*) are computed assuming a constant network size, while only outputs increase (increase in density over a fixed network), on the contrary returns to scale (*SCALE*) are computed with respect to an equi -proportional increase in both outputs and networks' size.

Given a multi outputs cost function, returns to density are given by:

$$DENSITY = \left(\sum_i \frac{\partial \ln(C)}{\partial \ln(q_i)} \right)^{-1}$$

While returns to scale are given by:

$$SCALE = \left(\sum_i \frac{\partial \ln(C)}{\partial \ln(q_i)} + \sum_i \frac{\partial \ln(C)}{\partial \ln(n_i)} \right)^{-1}$$

where C is total cost, q_i is the *i*th output and n_i is the network size associated to the provision of the *i*th output. The derivatives need to be interpreted as cost elasticities with respect to the *i*th output / network.

We can also estimate product specific economies of density and product specific economies of scale. The general formula for product *i* are:

$$DENSITY_i = \frac{(C(q_1, q_2, \ldots, q_m; \overline{p}) - C(q_1, q_2, \ldots, q_{i-1}, 0, q_{i+1}, ..., q_m; \overline{p}))/q_i}{\dfrac{\partial C}{\partial q_i}}$$

$$SCALE_i = \frac{(C(q_1, q_2, \ldots, q_m; \overline{p}) - C(q_1, q_2, \ldots, q_{i-1}, 0, q_{i+1}, \ldots, q_m; \overline{p})) / q_i}{\frac{\partial C}{\partial q_i} + \frac{\partial C}{\partial n_i}}$$

where the numerator is the average incremental cost (*AIC*), i.e. the additional costs of increasing product *i* from zero to q_i, holding all other outputs (and the input prices) fixed; while the denominator is the marginal cost for product *i*.

Economies of density / scale arise when *DENSITY* / *SCALE* is greater than one, while diseconomies of density / scale are present if *DENSITY* / *SCALE* is smaller than one. Neither economies nor diseconomies exist if *DENSITY* / *SCALE* is equal to one.

3. Data description

The dataset is an unbalanced panel covering annual information about 67 bus companies over the period 1998-2004. All firms are located in Piedmont, a region in Northern Italy and data were collected by the administrative offices of the local regional government[5]. The choice of a regional extent is particularly relevant because of its consistency with the Italian regulatory framework issued from the LPT reform process started with Law 549/1995, which transferred infrastructures and organizational resources to the local authorities corresponding to the Italian regions. The sample represents about ninety percent of the local public transport bus companies in Piedmont.

The variables used in the estimation of the cost function are total costs, output measures and input prices.

Table 1 shows some summary statistics for the whole sample while table 2 reports median levels, for outputs only, breaking the sample into publicly and privately owned bus companies.

[5] The data source is Conto Nazionale Trasporti (National Transportation Account, CNT)

Table1. Descriptive statistics: unbalanced panel, 67 firms, 411 observations over the period 1998-2004

Variable description	Variable names	Mean	Std. dev.	1st quartile	Median	3rd quartile
Total costs (th. Euro)	C	5,479	25,829	240	920	2,414
Labour costs (th. Euro)	LC	3,099	15,139	125	439	1,349
Capital costs (th. Euro)	KC	2,380	10,700	128	492	1,216
Labour share	SL	0.51	0.09	0.46	0.52	0.58
Labour price (th. Euro per employee)	p_L	28.03	6.73	24.39	28.45	31.29
Capital price (th. Euro per vehicle)	p_K	31.24	12.22	22.29	30.79	37.09
N. employees	EMP	97	450	6	16	47
N. vehicles	VEH	50	166	6	16	39
Vehicle / Km urban (th.)	q_U	2,213	8,724	72.61	180.5	865.9
Vehicle / Km intercity (th.)	q_I	1,111	2,080	118.6	326.7	1055.4
Network urban (Km)	n_U	125.30	199.67	17.20	33	135
Network intercity (Km)	n_I	465.57	946.63	63.65	182.8	440

Notes:

- The summary statistics for vehicle / kilometres and network length are computed with respect to the companies with values different from zero.
- All monetary variables are deflated using the consumer price index (source: ISTAT) using 1998 as the base year.

Table 2. Median output and network

	All	Public	Private
Vehicle / Km urban (th.)	180.5	876.6	113.6
Vehicle / Km intercity (th.)	326.7	549.9	279.8
Network urban (Km)	33	108.9	18.8
Network intercity (Km)	182.8	164.6	208.5

Total costs (C) are obtained as the sum of total costs associated to the production of urban and intercity services.

The cost function in equation (1) is a “multi output transport cost function” (see Jara-Dìaz and Cortés, 1996) because it is specified in terms of a vector representing several dimensions of the transport product: firstly transport product is represented by the total number of vehicle / kilometres, secondly the characteristics of the network are also introduced.

Vehicle / kilometres equal the product of the number of vehicles by the total number of kilometres covered over the year. We are also able to distinguish among vehicle/

kilometres covered over the urban network (q_U) from those associated to intercity connections (q_I) and this distinction is exploited in the empirical specification of the cost function for the derivation of scope economies. On average, firms that supply urban services, cover about 2.2 mil. vehicle / kilometres per year. This number halves when considering intercity services (1.1 mil. vehicle / km). The median number of vehicle / kilometres is always lower than the corresponding average, pointing to the fact that the sample distribution is quite asymmetric with a large number of small firms producing only small amounts of outputs on one side and a few very large firms (note that even the number of vehicle / kilometres at the 3rd quartile is smaller than the mean). Table 2 clarifies that the main source of such structure is the size difference among public (mainly municipal companies) and private bus companies[6].

The sample composition with respect to the output mix and the ownership is presented in table 3. 11 out of the 67 bus companies in the sample are publicly owned and most of them supply both urban and intercity services. On the contrary private firms are mainly specialised in the provision of intercity connections (32 out of 56 private firms) even if about 40% of them supply both services.

Table 3. Number of firms by product mix and ownership

Type of activity	All	Public	Private
Only urban	3	1	2
Only intercity	33	1	32
Both	31	9	22
Total number of firms	67	11	56

The second variable that enters the vector describing the transport output is the network length. Again we are able to distinguish among the urban network (n_U) and the intercity network (n_I). As expected the urban network is always smaller than the intercity network, whether we consider the average, median or any other quartile measures (see table 1). The median size of the urban network is particularly small for private firms, while the median size for intercity networks is comparable across firms with different ownership structures (see table 2).

[6] The largest company in the dataset is the public firm GTT (Gruppo Torinese Trasporti), owned by the municipality of the city of Turin. Its average number of vehicle kilometres is 45.7 mil. and 12.1 mil. for urban and intercity services respectively while its average number of employees is 3,400.

We consider only two input prices: the price of labour (p_L) and the price of other factors that we broadly indicate as capital price (p_K).

The cost of labour equals the sum of the total cost of labour associated to urban and intercity services, while the cost of capital is obtained as a residual measure, subtracting total labour cost from total costs. The price of labour is finally obtained dividing the cost of labour by the total number of employees (drivers and administrative staff), while the price of capital is given by the ratio of total cost of capital to the total number of vehicles (rolling stock of the bus company). The labour price does not show high variability in our sample: the median and the average values almost coincide for a yearly labour price per employee approximately equal to Euro 28,000. The capital price is more volatile and the median is Euro 30,790 per vehicle per year. The labour and capital shares are almost identical: labour costs represent on average 51% of total costs.

A linear time trend (*Trend*) and a quadratic time trend (*Trend2*) are also included in the final estimated specification. They should capture the effect of technological change over time.

All variables (except for time trend and total costs) are normalised by their sample median before estimation. Moreover total costs and labour price are divided by the capital price. The regularity conditions require that the cost function in (1) be nondecreasing in input prices and output, and linearly homogeneous and concave in input prices. The linear homogeneity condition is imposed by normalising total costs and labour price by capital price, while the other requirements (non decreasing costs in outputs and inputs and concavity in input prices) are checked after estimation.

4. Results

4.1 Estimation results

Table 4 reports the estimation results from the four specifications introduced in section 2: the standard translog (*Std. translog* column), the generalized translog (*Gen. translog*), the separable quadratic (*Sep. quadratic*) and the composite (*Composite*).

The first order parameters associated to the two outputs and labour price are always precisely estimated and have the expected positive signs. The urban network measure is significantly different from zero only in the generalized translog specification, while the coefficient for the intercity network is significant only in the last two specifications and it enters with the wrong negative sign in the standard translog model. Time trends are never significantly different from zero.

Table 4. Estimation results. Dependent variable: natural logarithm of total costs normalised by the capital price. Cluster robust standard errors in parenthesis. Number of observations 411.. Subscripts U and I for urban and intercity services respectively.

	Std. translog	*Gen. translog*	*Sep. quadratic*	*Composite*
q_U	0.341***	0.272***	396.771***	380.062***
	(0.06)	(0.03)	(63.91)	(59.90)
q_I	0.573***	0.506***	527.345***	528.881***
	(0.09)	(0.04)	(28.91)	(29.25)
q_U2	0.025***	0.017	3.058	2.637
	(0.00)	(0.02)	(4.38)	(4.13)
q_I2	0.055*	-0.151***	9.591	9.714
	(0.03)	(0.03)	(12.27)	(12.24)
q_U*q_I	-0.020**	-0.015	9.942	5.329
	(0.01)	(0.04)	(28.08)	(26.34)
$q_U*ln(p_L)$	0.012**	-0.167**		-83.496
	(0.00)	(0.06)		(105.42)
$q_I*ln(p_L)$	-0.084	0.035		34.905
	(0.10)	(0.10)		(151.54)
Trend	-0.014	-0.017	-0.037	-2.203
	(0.02)	(0.02)	(5.95)	(5.07)
Trend2	0.003	0.004	0.268	0.824
	(0.00)	(0.00)	(1.16)	(1.24)
$ln(p_L)$	0.687***	0.813***	0.562***	0.545**
	(0.15)	(0.09)	(0.06)	(0.20)
$ln(p_L)^2$	0.432**	0.031	0.117	0.055
	(0.21)	(0.23)	(0.17)	(0.17)
n_U	0.095	0.111***	36.275	39.733
	(0.07)	(0.03)	(40.71)	(49.40)
n_U2	0.007	-0.026	3.696	1.429
	(0.01)	(0.02)	(10.87)	(10.74)
q_U*n_U	0.000	0.030	-11.527	-8.436
	(0.00)	(0.02)	(11.84)	(10.94)
q_I*n_U	0.006	-0.090***	-4.594	-11.812
	(0.01)	(0.03)	(53.42)	(47.11)
$n_U*ln(p_L)$	-0.002	0.247**		46.120
	(0.01)	(0.08)		(54.19)
n_I	-0.049	0.036	30.710*	29.866*
	(0.09)	(0.05)	(17.41)	(17.24)
n_I^2	-0.163***	-0.016	-1.128*	-1.093*
	(0.03)	(0.01)	(0.62)	(0.61)
q_U*n_I	0.003	-0.070**	-17.146	-14.928
	(0.01)	(0.02)	(36.91)	(35.86)

$q_E * n_I$	0.046**	0.069**	-18.823	-18.498
	(0.02)	(0.02)	(13.28)	(13.20)
$n_I * ln(p_L)$	0.143	0.023		-0.548
	(0.13)	(0.11)		(0.60)
$n_U * n_I$	-0.023	0.141**	124.793	142.241
	(0.02)	(0.06)	(117.77)	(116.42)
Constant	7.359***	7.140***	63.520**	67.494***
	(0.08)	(0.06)	(21.01)	(15.88)
π	0	0.377***	1	1
		(0.03)		
R2adj	0.958	0.965	0.999	0.999
Log. Likelihood	-58.560	-19.506	45.883	47.977
RSS	31.998	26.460	19.249	19.053
AIC	163.121	87.012	-55.766	-51.955
BIC	255.548	183.458	16.569	36.454
Output regularity violations (n. points)	24	29	24	24
Price regularity violations (n. points)	0	7	0	2
LR test	78.11 ; p-value(1)=0.000		4.19 ; p-value(4) = 0.381	

Notes :

- All models are estimated using the non linear least squares routine "nl" in Stata 9.2. Many starting values have been provided and results did not display sensitivity to the chosen starting values. The displayed estimates are obtained starting from a vector of pseudorandom draws from a uniform [0,1) distribution.
- In the estimation of the standard translog specification, zero outputs level are substituted by the value 0.00001
- Standard errors are robust to heteroschedasticity of unknown form and to the likely presence of intra cluster correlation. Each cluster is represented by a different firm. Number of clusters 67 in all specifications.
- R2adj is the centered adjusted R2, Log. Likelihood is the value of the log-likelihood function, assuming errors are iid normal, while RSS is the residual sum of squares
- AIC and BIC are the Akaike's and Schwarz's Bayesian information criteria respectively
- Regularity violations are evaluated with respect to first order derivatives only: number of points where marginal cost with respect to output and labour price respectively are negative.
- LR test displays the statistics and the p – value (degrees of freedom in parenthesis) associated to two likelihood ratio tests (distributed as a Chi squared under the null). In the first case the null hypothesis is the validity of the restrictions of the standard translog with respect to the generalized translog (H0: $\pi = 0$); in the second case the null hypothesis is the validity of the restrictions of the separable quadratic with respect to the composite specification (H0: $q_U * ln(p_L) = q_I * ln(p_L) = n_U * ln(p_L) = n_I * ln(p_L) = 0$).
- Significance levels: * 10%; ** 5%; *** 1%.

Only the standard translog specification is a linear specification and the parameters can be interpreted as elasticities since all variables are expressed in logarithmic form. However labour price can be interpreted as the estimated labour share in all specifications (since it always enters linearly and in logarithmic form). While the two translog specifications give very high estimates for the labour shares (ranging from 0.69

to 0.81), the estimates from the separable quadratic (0.56) and the composite models (0.55) are closer to the actual median labour share (0.52).

The choice of the preferred specification may be driven by the best statistical fit or by the satisfaction of the regularity conditions of the cost function. Pulley and Braunstein (1992) highlight the potential trade off between the two choice strategies: "the more flexible the function is made to improve fit, the less likely it is that regularity conditions on derivatives will be satisfied". Researchers such as Pulley and Braunstein (1992) give preference to the statistical fit of the model, preferring the model with the highest log likelihood or the lowest value for the information criterion, others such as Barnett and Lee (1985) would choose the model that satisfy the regularity conditions over the largest number of observation points.

Based on the statistical fit the composite specification has the largest log likelihood and the smallest Akaike information criterion. However when performing a likelihood ratio tests on the restrictions imposed by the separable quadratic model (the four interaction terms among labour price and the two outputs and the two networks), the null hypothesis cannot be rejected[7] pointing to the validity of the separable quadratic specification that also displays the smallest Bayesian information criterion[8].

Table 4 also shows some clues about the degree of regularity conditions violations in our sample reporting the number of points where decreasing costs in outputs and inputs are uncovered. Both the separable quadratic and the composite cost functions do not do worse than the two translog models and this furthermore supports the choice of the quadratic specifications.

4.2. Scope and scale economies

Table 5 reports the estimated scope, density and scale economies at the sample median for the four specifications.

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[7] Moreover the four coefficients that are constrained to be zero in the separable quadratic model ($q_U*ln(p_L)$, $q_I*ln(p_L)$, $n_U*ln(p_L)$ and $n_I*ln(p_L)$), are not significantly different from zero in the composite model.

[8] The Akaike's and the Schwarz's Bayesian Information criteria are alternative measures of statistical fit that are often employed for the comparison of non nested models. The model with the lowest information criterion is the one ensuring the highest statistical fit.

As expected scope economies greatly differ across the models, while scale and density economies seem to converge on similar magnitudes.

For scope economies, an estimated value greater than zero indicates the presence of cost savings from the joint production of urban and intercity services, while the economies of scale / density are detected by magnitudes larger than 1.

The interpretation of scope economies is quite straightforward: they represent the percent difference between the total costs associated to a disjoint production and the total costs associated to a joint production. The four specifications give conflicting results: according to the two translog specifications diseconomies of scope are present (-0.94 from the standard translog and -0.35 from the generalized translog), while mild (but imprecisely) estimated scope economies are obtained from the other two models (0.02, standard errors approximately equal to 0.03).

Table 5 also reports density and scale economies. While scope economies are related with the addition of new products to the production set, the concepts of density and scale economies convey information about the behaviour of the cost function when more of each output in the existing production set is produced.

Global density economies range between 1.09 and 1.29: they are always high (especially in the last three specifications) and significantly different from 1. Global scale economies are smaller, but significantly different from 1 in all specifications (ranging from 1.04 and 1.08). We also report product specific density and scale economies as well cost elasticities with respect to output and network length. Density returns are always higher for urban services, while product specific scale returns are much similar across types of activities.

Many studies have computed density and scale economies for the LPT industry. Farsi et al. (2007) find scale economies for the median LPT Swiss firm equal to 1.11 while Ottoz et al. (2007) compute scale economies for a sample of Italian bus companies that range between 1.07 (for private firms) and 1.15 (for publicly owned companies). In the sample of large Italian bus companies Piacenza (2006) and Fraquelli et al. (2004) find large long run scale economies (1.86), while Cambini and Filippini (2003) find scale economies that differ according to the type of activity (1.17 for firms operating in urban and regional areas, 1.21 for companies operating in regional area and 1.29 for firms in urban areas). Piacenza (2001) and De Borger et al. (2002) are recent surveys on public transit companies studies, reporting a number of results with respect to scale and density economies.

Table 5. Global scope, density, scale economies and output and network cost elasticities evaluated at sample median.

	Std.Translog		Gen.Translog		Sep.Quadratic		Composite	
	Estimate	Std.Err.	Estimate	Std.Err.	Estimate	Std.Err.	Estimate	Std.Err.
SCOPE	-0.943	0.015	-0.352	0.069	0.017	0.035	0.019	0.033
DENSITY	1.094	0.096	1.285	0.068	1.204	0.076	1.229	0.073
SCALE	1.042	0.044	1.081	0.074	1.039	0.032	1.040	0.030
$DENSITY_U$	2.767	0.485	2.087	0.263	1.251	0.129	1.272	0.134
$DENSITY_I$	1.743	0.280	1.550	0.139	1.145	0.223	1.158	0.132
$SCALE_U$	0.655	0.018	1.481	0.083	1.018	0.014	1.014	0.014
$SCALE_I$	1.907	0.103	1.448	0.052	1.033	0.091	1.025	0.013
Elast. q_U	0.341	0.065	0.272	0.028	0.349	0.046	0.338	0.045
Elast. q_I	0.573	0.092	0.506	0.039	0.481	0.051	0.476	0.046
Elast. n_U	0.095	0.070	0.111	0.026	0.080	0.036	0.086	0.036
Elast. n_I	-0.049	0.091	0.036	0.054	0.052	0.055	0.062	0.054

Notes:
- In the estimation of the scope economies for the standard translog specification, zero outputs levels are substituted by the value 0.00001

Table 6 reports the estimated scope, density and scale economies at different sample points using the composite specification. We are going to assume the composite as our preferred model, however it should be pointed out that also the separable quadratic performs well in terms of the statistical fit to the data and the point estimates from the two specifications are very similar.

The median private firm shows higher scope, density and scale economies with respect to the median public firm. Scope economies for public firms are negative but not significantly different from one, pointing to absence of scope economies and scope diseconomies for the median public company.

As expected density and scale returns decrease with the firm size (global scale economies for the first quartile are equal to 1.16 while for the third quartile they are 0.97) and a similar pattern is found for scope economies and product specific density and scale returns.

Table 6. Global scope, density, scale economies and output and network cost elasticities evaluated at different sample points using the estimation results from the composite specification

	1st quartile		3rd quartile		Median public		Median private	
	Estimate	Std.Err.	Estimate	Std.Err.	Estimate	Std.Err.	Estimate	Std.Err.
SCOPE	0.130	0.034	-0.086	0.071	-0.012	0.042	0.044	0.036
DENSITY	1.326	0.096	1.283	0.193	1.147	0.071	1.247	0.065
SCALE	1.159	0.044	0.970	0.058	1.027	0.045	1.069	0.035
$DENSITY_U$	1.434	0.203	1.209	0.171	1.147	0.106	1.275	0.132
$DENSITY_I$	1.352	0.507	1.108	0.066	1.188	0.282	1.149	0.107
$SCALE_U$	1.047	0.031	1.011	0.068	0.986	0.013	1.011	0.087
$SCALE_I$	1.005	0.004	0.973	0.084	1.058	0.032	0.901	0.137
Elast. q_U	0.333	0.048	0.394	0.062	0.586	0.055	0.275	0.040
Elast. q_I	0.421	0.032	0.385	0.137	0.286	0.073	0.527	0.041
Elast. n_U	0.067	0.048	0.140	0.059	0.053	0.062	0.073	0.028
Elast. n_I	0.042	0.026	0.112	0.144	0.049	0.068	0.060	0.047

As pointed out by Pulley and Humphrey (1993), scope economies may arise from two sources of cost savings: the possibility to reduce excess capacity reaching smaller fixed costs and the existence of production complementarities that allow for smaller production costs.

The difference between the two components corresponds to the possibility to save on fixed costs versus variable costs when jointly producing more than one output.

For the case of bus companies, fixed costs savings may be associated to the ability to reduce excess capacity by sharing vehicles, offices, parking areas, bus garage and maintenance centres, etc. Costs complementarities are present if some variable inputs can be shared by different product lines: e.g. drivers and administrative staff, fuels and bus parts and accessories could in principle be used for both urban and intercity services, with cost savings arising for example from discounts on large quantity purchases or from the possibility to shift workers across services according to daily needs.

The composite model, unlike the translog cost function allows for the distinction between these two sources of costs savings (see the derivation in Pulley and Humphrey, 1993). Table 7 reports the results for a number of sample points. Global scope economies are simply given by the sum of scope economies from fixed costs savings and scope economies from cost complementarities (the two columns in table 7).

For the median firm, our estimated global scope economies amount to 1.9%, not significantly different from zero, but table 7 makes it evident that such a number should be split into cost savings from fixed costs equal to 6.3% (joint production allows for the reduction of excess capacity) which is statistically significant and diseconomies equal to -4.4% arising from cost complementarities (only weakly significantly different from zero).

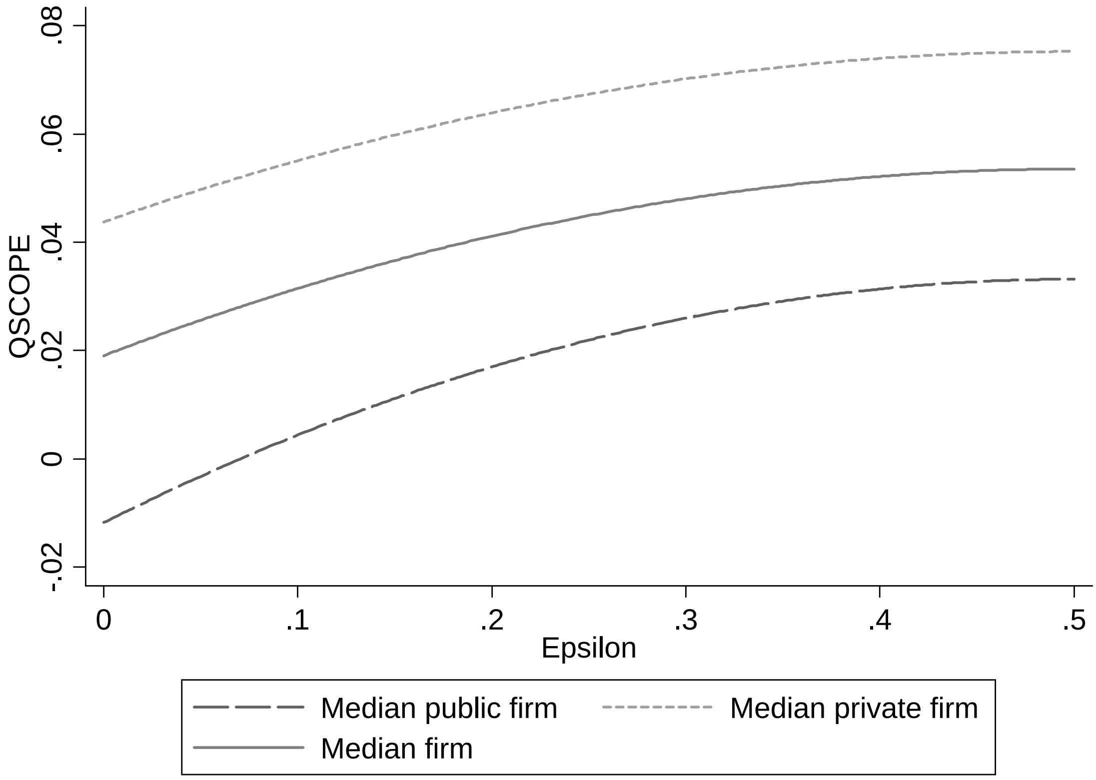
Table 7. Estimates of fixed costs and cost complementarities effects at different sample points using the estimation results from the composite specification (asymptotic standard errors in parenthesis).

	Scope economies from fixed costs	Scope economies from cost complementarities
1st quartile	0.150 (0.031)	-0.019 (0.011)
Median firm	0.063 (0.014)	-0.044 (0.027)
3rd quartile	0.017 (0.004)	-0.103 (0.069)
Median Public firm	0.023 (0.005)	-0.035 (0.041)
Median Private firm	0.081 (0.018)	-0.038 (0.029)

A similar pattern is observed also for the first and third quartile and for the median public and private firms. The smaller is the firm, the greater the cost savings from fixed costs and the less important the diseconomies from cost complementarities. The only exception is the median public firm that displays high diseconomies from cost complementarities and small economies from fixed costs savings.

Figure 1 shows the estimated quasi scope economies as a function of ε (see section 2.2 for the definition). The pattern of quasi scope economies corroborates the results about the larger scope economies resulting from fixed costs savings and excess capacity reduction. In fact quasi scope economies raise as ε increases from 0 to ½. As stand alone productions become less specialised, cost advantages increase. In particular when ε = ½, *QSCOPE* is measuring the percent difference in costs among two firms supplying one half of urban service and one half of intercity connections each and the total cost of a single firm supplying both urban and intercity services to the whole market. In this case the concept of specialization disappears: we are not any more comparing specialised and joint productions, but firms that are operating at different scale levels. In this case *QSCOPE* is a measure of the fixed costs effects (the possibility that joint production can allow for cost savings thanks to reduced excess capacity) and scale economies. The fact that the *QSCOPE* increase as ε increases confirms the key role that is played by the fixed costs component versus the cost complementarities component of scope economies.

Figure 1. Quasi scope economies for different values of ε (*Epsilon) using the estimation results from the composite specification*



5. Concluding remarks

The study analyses the density, scale and scope economies for a sample of Italian bus companies estimating a total cost function. Most of the firms either supply only intercity services (33 companies out of 67) or offer both urban and intercity connections (31 companies). Our main interest is in the presence of sizeable cost savings from the joint production of urban and intercity activities.

In an attempt to overcome the drawbacks of the standard translog model that is unable to handle zero outputs levels and whose behaviour in the neighbourhood of zero outputs is degenerate (see Roller, 1990; Pulley and Braunstein, 1992), we present a number of different cost specifications that mainly differ in the way outputs are introduced into the functions. The separable quadratic and the composite models have quadratic structures in outputs and log-quadratic structures in input prices. The quadratic output structure was first recommended by Baumol et al. (1982) when examining scope economies because this form allows for the direct handling of zero outputs, without any need for substitutions or transformations as in the translog models.

The estimated scope economies largely differ across the specifications and, as expected, the translog models imply very large scope diseconomies, that are not reliable.

Using the composite functional form, results point to the presence of only moderate scope economies. However large cost advantages can be obtained from fixed costs savings, i.e. it seems that there is some excess capacity among the bus companies in our sample and the joint production of urban and intercity services can allow for the full exploitation of the available fixed inputs.

Given the availability of data on network length, we are also able to distinguish among density and scale economies for the bus companies in our sample. Density returns are high, while scale economies are smaller in size, but always significantly greater than one. Lower average costs can be attained from an increase in the size of the firms, both in terms of covered vehicle – kilometres (our output measure) and in terms of network length.

References

Adams R.M., Bauer P.W. and Sickle R.C., 2004, 'Scale economies, scope economies and technical change in Federal Reserve payment processing', *Journal of Money Credit and Banking*, **36**, 943-958

Barnett W.A. and Lee Y.W., 1985, 'The global properties of the Miniflex Laurent, Generalized Leontief and Translog Flexible functional forms', *Econometrica*, **53**, 1421-1437

Baumol W.J., Panzar J.C. and Willig R.D., 1982, *Contestable markets and the theory of industry structure*, New York, Harcourt, Brace, Jovanovich, Inc.

Bloch H., Madden G., Savage S. J., 2001, 'Economies of scale and scope in Australian telecommunications', *Review of Industrial Organization*, 18, 219-227

Box G. and Cox D., 1964, 'An analysis of transformations', *Journal of the Royal Statistical Society*, Series B, **26**, 211-246

Cambini C. and Filippini M., 2003, 'Competitive tendering and optimal size in the regional bus transportation, industry. An example from Italy', *Annals of Public and Cooperative Economics*, **74**(1), 163-82

Cameron A.C. and Trivedi P.K., 2005, *Microeconometrics. Methods and Applications*, New York: Cambridge University Press

Carey K., 1997, 'A panel data design for estimation of hospital cost functions', *Review of Economics and Statistics*, **79** (3), 443-453

Caves D.W., Christensen L.R. and Tretheway M.W., 1980, Flexible cost functions for multiproduct firms, *Review of Economics and Statistics*, **62**, 477-481

Caves D.W., Christensen L.R. and Tretheway M.W., 1984, Economies of density versus economies of scale: why trunk and local airline costs differ, *Rand Journal of Economics*, **15**, 471-489

Christensen L.R., Jorgenson D.W. and Lau L.J., 1971, 'Conjugate duality and the transcendental logarithmic production function', *Econometrica*, **39**, 255-256

De Borger B., Kerstens K. and Costa A., 2002, 'Public transit performance: what does one learn from frontier studies?', *Transport Reviews*, **22**(1), 1-38

Diewert W.E., 1974, ‘Applications of duality theory’, in ‘*Frontiers in Quantitative Economics*’, Vol 2 (Eds.) M. D. Intriligator and D. A. Kendrick, North Holland, Amsterdam, pp. 106– 71

Diewert W.E. and Wales T.J., 1987, ‘Flexible functional forms and global curvature conditions’, *Econometrica*, **55**(1), 43-68

Farsi M., Fetz A. and Filippini M., 2007, ‘Economies of scale and scope in local public transportation’, *Journal of Transport Economics and Policy*, forthcoming

Fraquelli G., Piacenza M. and Abrate G, 2004a, ‘Regulating Public Transit Systems: How Do Urban-Intercity Diversification and Speed-up Measures Affect Firms’ Cost Performance?’, *Annals of Public and Cooperative Economics*, **75**(2), 193-225

Fraquelli G., Piacenza M. and Vannoni D., 2004b, ‘Scope and scale economies in multi-utilities: evidence from gas, water and electricity combinations’, *Applied Economics*, **36**, 2045-2057

Fraquelli G., Piacenza M. and Vannoni D., 2005, ‘Cost Savings from Generation and Distribution with an Application to Italian Electric Utilities’, *Journal of Regulatory Economics*, **28**(3), 289-308

Garcia S. and Thomas A., 2001, ‘The structure of municipal water supply costs: application to a panel of French local communities’, *Journal of Productivity Analysis*, **16**, 5-29

Greene W., 2001, ‘Estimating Econometric Models with Fixed Effects’, mimeo

Jara-Dìaz S.R. and Cortés C.E., 1996, ‘On the calculation of scale economies from transport cost functions’, *Journal of Transport Economics and Policy*, **30**(2), 157-170

McKillop D.G., Glass J.C. and Morikawa Y., 1996, ‘The composite cost function and efficiency in giant Japanese banks’, *Journal of Banking and Finance*, **20**, 1651-1671

Ottoz E., Fornengo G. and Di Giacomo M., 2007, ‘The impact of ownership on the cost of bus service provision: an example from Italy’, *Applied Economics*, forthcoming

Piacenza M., 2001, ‘Productive Structure, Cost Efficiency and Incentives in the Local Public Transport: A Survey of Theoretical and Empirical Issues’, *working paper Hermes*

Piacenza M., 2006, ‘Regulatory Contracts and Cost Efficiency: Stochastic Frontier Evidence from the Italian Local Public Transport’, *Journal of Productivity Analysis*, forthcoming

Piacenza M. and Vannoni D., 2004, ‘Choosing Among Alternative Cost Function Specifications: An Application to Italian Multi-Utilities’, *Economics Letters*, **82(**3), 410-417

Pulley L.B. and Braunstein Y.M., 1992, ‘A composite cost function for multiproduct firms with an application to economies of scope in banking’, *Review of Economics and Statistics*, **74**, 221-230

Pulley L.B. and Humphrey D.B., 1993, ‘Scope economies: Fixed costs, complementarity and functional form’, Journal of Business, **66**(3), 437-462

Roller L.-H., 1990, ‘Proper quadratic cost functions with an application to the Bell System’, *Review of Economics and Statistics*, **72**, 202-210.

Appendix

This appendix presents the formulas for the four cost function specifications presented in the paper and some results with respect to marginal costs and cost elasticities.

In the following we are going to exploit the fact that:

$$\frac{\partial \ln(C)}{\partial \ln q_i} = \frac{\partial C}{\partial q_i}\frac{q_i}{C} \quad \text{and} \quad \frac{\partial \ln(C)}{\partial \ln q_i}\frac{C}{q_i} = \frac{\partial C}{\partial q_i}$$

deriving the right hand side or the left hand side of the formulas, depending on the specification.

Marginal costs, cost elasticities and scope economies are all evaluated keeping fixed, at their sample median level, the input prices. Since input prices enter all specifications in logarithmic form and all variables are normalised by their sample median level before estimation, it follows that they can be ignored since their value is equal to zero ($\ln(\overline{p_r} / \overline{p_r}) = \ln(1) = 0$, where the bar indicates the median level).

1. The translog specification

$$\ln(C) = \Big\{ \alpha_0 + \sum{}_i \alpha_i \ln q_i + \frac{1}{2}\sum{}_{i,j} \alpha_{ij} \ln q_i \ln q_j + \sum{}_{i,r} \alpha_{ir} \ln q_i \ln p_r +$$
$$+ \sum{}_i \delta_i \ln n_i + \frac{1}{2}\sum{}_{i,j} \delta_{ij} \ln n_i \ln n_j + \sum{}_{i,r} \delta_{ir} \ln n_i \ln p_r + \frac{1}{2}\sum{}_{i,j} \lambda_{ij} \ln q_i \ln n_j +$$
$$\sum{}_r \beta_r \ln p_r + \frac{1}{2}\sum{}_{r,k} \beta_{rk} \ln p_r \ln p_k \Big\}$$

$$\frac{\partial \ln(C)}{\partial \ln q_i} = \alpha_i + \alpha_{ij} \ln q_j + \alpha_{ii} \ln q_i + \lambda_{ij} \ln n_j + \lambda_{ii} \ln n_i$$

2. *The generalized translog model*

$$\ln(C) = \Big\{\alpha_0 + \sum_i \alpha_i q_i^{(\pi)} + \frac{1}{2}\sum_{i,j} \alpha_{ij} q_i^{(\pi)} q_j^{(\pi)} + \sum_{i,r} \alpha_{ir} q_i^{(\pi)} \ln p_r +$$

$$+ \sum_i \delta_i n_i^{(\pi)} + \frac{1}{2}\sum_{i,j} \delta_{ij} n_i^{(\pi)} n_j^{(\pi)} + \sum_{i,r} \delta_{ir} n_i^{(\pi)} \ln p_r + \frac{1}{2}\sum_{i,j} \lambda_{ij} q_i^{(\pi)} n_j^{(\pi)} +$$

$$\sum_r \beta_r \ln p_r + \frac{1}{2}\sum_{r,k} \beta_{rk} \ln p_r \ln p_k \Big\}$$

$$\frac{\partial \ln(C)}{\partial \ln q_i} = q_i^{\pi-1}(\alpha_i + \alpha_{ij} q_j^{(\pi)} + \alpha_{ii} q_i^{(\pi)} + \lambda_{ij} n_j^{(\pi)} + \lambda_{ii} n_i^{(\pi)})$$

3. *The separable quadratic function:*

$$\ln(C) = \Big\{\ln\Big(\alpha_0 + \sum_i \alpha_i q_i + \frac{1}{2}\sum_{i,j} \alpha_{ij} q_i q_j + \sum_i \delta_i n_i +$$

$$\frac{1}{2}\sum_{i,j} \delta_{ij} n_i n_j + \frac{1}{2}\sum_{i,j} \lambda_{ij} q_i n_j\Big) + \Big[\sum_r \beta_r \ln p_r + \frac{1}{2}\sum_{r,k} \beta_{rk} \ln p_r \ln p_k\Big]\Big\}$$

$$C = \Big\{\Big(\alpha_0 + \sum_i \alpha_i q_i + \frac{1}{2}\sum_{i,j} \alpha_{ij} q_i q_j + \sum_i \delta_i n_i + \frac{1}{2}\sum_{i,j} \delta_{ij} n_i n_j +$$

$$\frac{1}{2}\sum_{i,j} \lambda_{ij} q_i n_j\Big) \cdot \exp\Big[\sum_r \beta_r \ln p_r + \frac{1}{2}\sum_{r,k} \beta_{rk} \ln p_r \ln p_k\Big]\Big\} = h(\mathbf{q},\mathbf{n}) \cdot f(\mathbf{p})$$

$$\frac{\partial C}{\partial q_i} = \left(\alpha_i + \alpha_{ij} q_j + \alpha_{ii} q_i + \lambda_{ij} n_j + \lambda_{ii} n_i\right)$$

4. The composite model

$$\ln(C)=\left\{\ln\left(\alpha_0+\sum{}_i\alpha_i q_i+\frac{1}{2}\sum{}_{i,j}\alpha_{ij}q_i\,q_j+\sum{}_{i,r}\alpha_{ir}q_i\,\ln p_r+\right.\right.$$

$$\left.+\sum{}_i\delta_i n_i+\frac{1}{2}\sum{}_{i,j}\delta_{ij}n_i\,n_j+\sum{}_{i,r}\delta_{ir}n_i\,\ln p_r+\frac{1}{2}\sum{}_{i,j}\lambda_{ij}q_i\,n_j\right)+$$

$$\left.\left[\sum{}_r\beta_r\ln p_r+\frac{1}{2}\sum{}_{r,k}\beta_{rk}\ln p_r\ln p_k\right]\right\}$$

$$C=\left\{\left(\alpha_0+\sum{}_i\alpha_i q_i+\frac{1}{2}\sum{}_{i,j}\alpha_{ij}q_i\,q_j+\sum{}_{i,r}\alpha_{ir}q_i\,\ln p_r+\right.\right.$$

$$\left.+\sum{}_i\delta_i n_i+\frac{1}{2}\sum{}_{i,j}\delta_{ij}n_i\,n_j+\sum{}_{i,r}\delta_{ir}n_i\,\ln p_r+\frac{1}{2}\sum{}_{i,j}\lambda_{ij}q_i\,n_j\right)\cdot$$

$$\left.\exp\left[\sum{}_r\beta_r\ln p_r+\frac{1}{2}\sum{}_{r,k}\beta_{rk}\ln p_r\ln p_k\right]\right\}=h(\mathbf{q},\mathbf{n},\mathbf{p})\cdot f(\mathbf{p})$$

$$\frac{\partial C}{\partial q_i}=\left(\alpha_i+\alpha_{ij}q_j+\alpha_{ii}q_i+\lambda_{ij}n_j+\lambda_{ii}n_i\right)$$

$$SCOPE=\frac{\left((m-1)\alpha_o-\sum_{i,j\neq i}\alpha_{ij}q_iq_j-\sum_{i,j\neq i}\delta_{ij}n_in_j-\sum_{i,j\neq i}\lambda_{ij}q_in_j\right)}{h(\mathbf{q},\mathbf{n},\mathbf{p})}$$

$$SCOPE_{FixedCost}=\frac{\left((m-1)\alpha_o\right)}{h(\mathbf{q},\mathbf{n},\mathbf{p})}$$

$$SCOPE_{CostComple\ mentaritie\ s}=\frac{-\left(\sum_{i,j\neq i}\alpha_{ij}q_iq_j+\sum_{i,j\neq i}\delta_{ij}n_in_j+\sum_{i,j\neq i}\lambda_{ij}q_in_j\right)}{h(\mathbf{q},\mathbf{n},\mathbf{p})}$$